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On Classical Ferromagnets with a Complex External Field

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We prove analyticity properties of correlation functions using correlation inequalities.

KEY WORDS: Classical ferromagnets; complex external field.

1. INTRODUCTION

In this note we prove analyticity properties of the correlation functions in the external magnetic field h for some ferromagnetic one-component models, provided $|\mathrm{Re}\, h| > |\mathrm{Im}\, h|$. Our proof is closely related to the work of Dunlop[(1)] and is a simple consequence of our extension of the correlation inequalities of Ginibre,[(2)] which we used to study the translation invariant states of the planar rotor model.[(3)] Our results hold in particular for the Ising model. In this case Lebowitz and Penrose proved stronger results, valid for $|\mathrm{Re}\, h| > 0$.[(4)] We also show that the infinite volume limit of a state constructed with a boundary field $h_b > 0$ is independent of the magnitude of h_b and hence equal to the state obtained from + boundary condition. This result was already proved by Lebowitz for the Ising model.[(5)]

Some of our results, as well as other applications on the absence of symmetry breakdown of continuous symmetry in two-dimensional models, were announced in Ref. 6.

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2. BASIC INEQUALITIES

The main results of this section are the inequalities (2.16) and (2.17). They are summarized at the end of the section. We first consider the Ising model. Let Λ be a finite set. For each $x \in \Lambda$ we have a spin variable $s(x)$. In the absence of interactions the spins are independent random variables and $s(x)$ is distributed according to the probability measure

$$d\mu(s(x)) = \tfrac{1}{2}\left[\delta(s(x)+1) + \delta(s(x)-1)\right] \tag{2.1}$$

We introduce an external magnetic field $h(x)$ and ferromagnetic interactions described by a two-body potential $J(x, y) \geqslant 0$. Hence the Gibbs measure is given by the expression

$$Z^{-1}(\{h(x)\})\exp\left(\sum_{\{x,y\}\subset\Lambda} J(x,y)s(x)s(y) + \sum_{x\in\Lambda} h(x)s(x)\right)\prod_{x\in\Lambda} d\mu(s(x)) \tag{2.2}$$

where the inverse temperature is equal to one and $Z(\{h(x)\})$ is the partition function. Expectation values with respect to the measure (2.2) are denoted by $\langle\cdot\rangle(\{h(x)\})$. In order to study the correlation functions

$$\left\langle \prod_{x\in A} s(x)\right\rangle_\Lambda (\{h(x)\}), \qquad A\subset\Lambda \tag{2.3}$$

with complex external field $h(x)$ we introduce another copy of our system with the external field $\overline{h(x)}$ and we consider the product

$$\left\langle \prod_{x\in A} s(x)\right\rangle_\Lambda (\{h(x)\})\left\langle \prod_{x\in A} s'(x)\right\rangle_\Lambda (\{\overline{h(x)}\}) \tag{2.4}$$

We now define new variables by the relations

$$\tfrac{1}{2}(s(x)+s'(x)) = \cos\theta(x), \qquad \tfrac{1}{2}(s(x)-s'(x)) = \sin\theta(x) \tag{2.5}$$

In our particular case $\theta(x) = k(\pi/2)$, $k = 0, 1, 2, 3$. We notice that

$$s(x)s'(x) = \cos^2\theta(x) - \sin^2\theta(x) = \cos 2\theta(x) \tag{2.6}$$

and for $\operatorname{Re} h(x) = h_1(x) > |\operatorname{Im} h(x)| = |h_2(x)|$

$$\begin{aligned} h(x)s(x) + \overline{h(x)}s'(x) &= h_1(x)(s(x)+s'(x)) + ih_2(x)(s(x)-s'(x)) \\ &= 2h_1(x)\cos\theta(x) + 2ih_2(x)\sin\theta(x) \\ &= 2k(x)\left[\operatorname{ch}\lambda(x)\cos\theta(x) + i\operatorname{sh}\lambda(x)\sin\theta(x)\right] \\ &= 2k(x)\cos\left[\theta(x) + i\lambda(x)\right] \end{aligned} \tag{2.7}$$

where $k(x) > 0$ and $k^2(x) = h_1^2(x) - h_2^2(x)$. Using the variables $\theta(x)$ the product of the correlation functions in (2.4) can be written as the expectation value of

$$\prod_{x \in A} \cos 2\theta(x) \tag{2.8}$$

with respect to the Gibbs measure of a $\mathbb{Z}_4$ model, whose Boltzmann factor is

$$\exp\left\{ \sum_{\{x,y\} \subset \Lambda} 2J(x,y)\cos[\theta(x) - \theta(y)] + \sum_{x \in \Lambda} 2k(x)\cos[\theta(x) + i\lambda(x)] \right\} \tag{2.9}$$

Using Proposition 1 of Ref. 3 we find that the expectation value of (2.8) decreases if we set $\lambda(x) \equiv 0$ in (2.9). Therefore we get the inequality

$$\left| \left\langle \prod_{x \in A} s(x) \right\rangle_\Lambda (\{h(x)\}) \right| \geqslant \left\langle \prod_{x \in A} s(x) \right\rangle_\Lambda (\{k(x)\}) \tag{2.10}$$

Similarly we have for the partition function

$$|Z(\{h(x)\})| \geqslant Z(\{k(x)\}) \tag{2.11}$$

which implies the result of Dunlop on the zeros of partition functions.[1]

Our results are of course valid for any distribution $d\mu(s)$ for which we can apply the method of Griffiths.[7] In such a case one can express $s(x)$ as a sum of Ising variables $\sigma_i(x) = \pm 1$, $i = 1, \ldots, N$ as

$$s(x) = \sum_{i=1}^{N} \sigma_i(x) \tag{2.12}$$

and the probability that $s(x) = t$, computed with respect to $d\mu(s)$, is equal to

$$\left(\sum_{\sigma_1, \ldots, \sigma_N} \exp \sum_{i,j} a_{ij}\sigma_i\sigma_j \right)^{-1} \left(\sum_{\substack{\sigma_1, \ldots, \sigma_N \\ \sum \sigma_i = t}} \exp \sum_{i,j} a_{ij}\sigma_i\sigma_j \right) \tag{2.13}$$

where $a_{ij} \geqslant 0$. Introducing the variables

$$\tfrac{1}{2}[\sigma_i(x) - \sigma_i'(x)] = \cos\theta_i(x), \qquad \tfrac{1}{2}[\sigma_i(x) + \sigma_i'(x)] = \sin\theta_i(x) \tag{2.14}$$

and using

$$s(x)s'(x) = \sum_{i=1}^{N} \cos 2\theta_i(x) + \sum_{i>j} 2\cos[\theta_i(x) + \theta_j(x)] \tag{2.15}$$

we can prove as before the inequality

$$\left| \left\langle \prod_{x \in A} s(x)^{m(x)} \right\rangle_\Lambda (\{h(x)\}) \right| \geqslant \left\langle \prod_{x \in A} s(x)^{m(x)} \right\rangle (\{k(x)\}) \tag{2.16}$$

where $m(x)$ are arbitrary positive integers. Similarly

$$|Z(\{h(x)\})| \geqslant Z(\{k(x)\}) \tag{2.17}$$

By a limiting procedure we see that we can take for $d\mu(s)$

$$d\mu(s) = \chi_{[-1,1]}(s)\, ds \tag{2.18}$$

Our results are also valid for the class of measures $d\mu(s)$ considered by Dunlop in Ref. 1. In this class we have for example the measures

$$d\mu(s) = \exp\left(-\lambda_6 s^6 - \lambda_4 s^4 - \lambda_2 s^2\right) ds \tag{2.19}$$

with $\lambda_6 \geqslant 0$, $\lambda_4 \geqslant 0$, and λ_2 real and

$$d\mu(s) = \exp\left[-ks^{2n} + P(s)\right] ds \tag{2.20}$$

with $P(s)$ any even polynomial with positive coefficients, $2n > \deg P$ and $k > 0$. Letting n going to infinity we get the measure

$$d\mu(s) = \chi_{[-1,1]}(s) \exp\left[P(s)\right] ds \tag{2.21}$$

[Notice that the case (2.19) with $\lambda_6 = 0$ can be obtained with the method of Griffiths.[8]] Indeed if we introduce

$$\frac{1}{\sqrt{2}}\left[s(x) + s'(x)\right] = r(x)\cos\theta(x) \tag{2.22}$$

$$\frac{1}{\sqrt{2}}\left[s(x) - s'(x)\right] = r(x)\sin\theta(x) \tag{2.23}$$

then

$$s(x)s'(x) = \tfrac{1}{2}\left[r(x)\right]^2 \cos 2\theta(x) \tag{2.24}$$

and in the case (2.19)

$$\begin{aligned} d\mu(s)\, d\mu(s') = {} & \exp\left(-\lambda_6 r^6 - \lambda_4 r^4 - \lambda_2 r^2\right) \\ & \exp\left[\cos^2 2\theta\left(\tfrac{1}{2}\lambda_4 r^4 + \tfrac{3}{4}\lambda_6 r^6\right)\right] r\, dr\, d\theta \end{aligned} \tag{2.25}$$

From this we get again (2.16) and (2.17) using a straightforward extension of Proposition 1 of Ref. 3 (see the discussion of Model 2 in Ref. 2). The case (2.20) is obtained using the results of Dunlop.[1] Instead of using the variables $r(x)$, one used new variables $\tau(x)$, defined by

$$s(x)^{2n} + s'(x)^{2n} = \tau(x)^{2n} \tag{2.26}$$

The variable τ can be expressed as

$$\tau^{2n} = r^{2n} \prod_{\substack{k=0 \\ k \text{ even}}}^{n-2} \left[1 - \cos^2 \frac{(k+1)\pi}{2n} \cos^2 2\theta\right] \tag{2.27}$$

or

$$r=\tau\prod_{\substack{k=0\\k\text{ even}}}^{n-2}\left[1-\cos^2\frac{(k+1)\pi}{2n}\cos^2 2\theta\right]^{-1/2n} \tag{2.28}$$

Therefore r can be expanded in a power series of the variable $\cos 2\theta$ with positive coefficients.

Finally we prove the following inequalities for $\mathrm{Re}\,h > |\mathrm{Im}\,h|$:

$$\mathrm{Re}\langle s(x)\rangle_\Lambda(\{h(x)\}) \geqslant \langle s(x)\rangle_\Lambda(\{k(x)\}) \tag{2.29}$$

$$\mathrm{Re}\langle s(x)s(y)\rangle_\Lambda(\{h(x)\}) \geqslant \langle s(x)s(y)\rangle_\Lambda(\{k(x)\}) \tag{2.30}$$

with $k(x)\geqslant 0$, $k^2(x)=[\mathrm{Re}\,h(x)]^2-[\mathrm{Im}\,h(x)]^2$, and

$$\mathrm{Re}\langle s(x)\rangle_\Lambda(\{h(x)\}) \geqslant |\mathrm{Im}\langle s(x)\rangle_\Lambda(\{h(x)\})| \tag{2.31}$$

For example, in the Ising model (2.31) follows from

$$\langle\cos[\theta(x)+i\eta(x)]\rangle_\Lambda(\{h(x)\}) \geqslant \langle\cos\theta(x)\rangle_\Lambda(\{k(x)\}) \tag{2.32}$$

in the duplicated system. But the left-hand side of (2.32) is equal to

$$\mathrm{ch}\,\eta(x)\mathrm{Re}\langle s(x)\rangle_\Lambda(\{h(x)\}) - \mathrm{sh}\,\eta(x)\mathrm{Im}\langle s(x)\rangle_\Lambda(\{h(x)\}) \tag{2.33}$$

Dividing by $\mathrm{ch}\,\eta(x)$ and letting $\eta(x)\to\pm\infty$ we get the result (2.31). By taking $\eta(x)=0$ we obtain (2.29). Inequalities (2.31) and the positivity of the left-hand side of (2.29) and (2.30) have already been proved by Dunlop.[9]

SUMMARY

We have proved for a large class of ferromagnetic one-component spin systems with two-body interactions, including all models with *a priori* measures given by (2.13), (2.19), and (2.21), that the correlation functions, and the partition functions, with a complex external magnetic field $h(x)$, $|\mathrm{Re}\,h(x)| > |\mathrm{Im}\,h(x)|$, satisfy the inequalities

$$\left|\left\langle\prod_{x\in A}s(x)^{m(x)}\right\rangle_\Lambda(\{h(x)\})\right| \geqslant \left\langle\prod_{x\in A}s(x)^{m(x)}\right\rangle_\Lambda(\{k(x)\}) \tag{2.34}$$

and,

$$|Z(\{h(x)\})| \geqslant Z(\{k(x)\}) \tag{2.35}$$

respectively, $k(x)>0$, $k^2(x)=[\mathrm{Re}\,h(x)]^2-[\mathrm{Im}\,h(x)]^2$.

3. APPLICATIONS

In this section Λ is a finite subset of the lattice $\mathbb{Z}^d$ and we assume that $J(x,y)=J(x-y)\geqslant 0$ is translation invariant and $\sum_x J(x)<\infty$. We also assume that the partition functions and the correlation functions are well defined for all Λ and all real magnetic fields $h(x)$.

In our first application we want to give a short proof that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\left\langle \prod_{x\in A}[s(x)]^{m(x)}\right\rangle_\Lambda(h)=\left\langle \prod_{x\in A}[s(x)]^{m(x)}\right\rangle(h) \tag{3.1}$$

is an analytic function of the external field $h(x)\equiv h$ for $|\mathrm{Re}\,h|>|\mathrm{Im}\,h|$ provided (2.34) and (2.35) are valid. Indeed, since we have free boundary conditions we know, by standard arguments using correlation inequalities, that the limit (3.1) exists and is positive if h is positive. Therefore if $h\in G_0=\{h\in\mathbb{C}:0<\mathrm{Re}\,h<h_0,\ \mathrm{Re}\,h>|\mathrm{Im}\,h|\}$, then

$$g_\Lambda(h)=\frac{1}{\langle\prod_{x\in A}[s(x)]^{m(x)}\rangle_\Lambda(h)} \tag{3.2}$$

is analytic in G_0 and from (2.34) we see that there exists a constant M such that

$$|g_\Lambda(h)|\leqslant M,\qquad \text{all }\Lambda,\quad h\in G_0 \tag{3.3}$$

The theorem of Vitali ensures that $\lim_{\Lambda\uparrow\mathbb{Z}^d}g_\Lambda(h)=g(h)$ exists and defines an analytic function of h in G_0. Since $g_\Lambda(h)\neq 0$ in G_0 and $g(h)>0$ for $h\in G_0\cap\mathbb{R}$, we have by Hurwitz theorem that $g(h)\neq 0$ for $h\in G_0$. This implies that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\left\langle \prod_{x\in A}[s(x)]^{m(x)}\right\rangle_\Lambda(h)=\frac{1}{g(h)} \tag{3.4}$$

is analytic for $h\in G_0$. The case with $\mathrm{Re}\,h<0$ is obtained by symmetry. Obviously the proof is valid for other boundary conditions provided we know the existence of the limit (3.1) for h real.

In our second application we consider only bounded spins with finite range interactions and constant external field $h\geqslant 0$. Without restricting the generality we suppose that

$$\sup\{s: s\in\operatorname{supp}\mu\}=1 \tag{3.5}$$

and the existence of $R\geqslant 1$ such that

$$J(x)>0\quad\text{if}\quad |x|=1\quad\text{and}\quad J(x)=0\quad\text{if}\quad |x|>R \tag{3.6}$$

Here $|x|^2=\sum_{i=1}^d|x_i|^2$. For any finite set $\Lambda\subset\mathbb{Z}^d$ we define

$$\partial_R\Lambda=\{x\in\Lambda:\mathrm{dist}(x,\mathbb{Z}^d\backslash\Lambda)\leqslant R\} \tag{3.7}$$

The + boundary condition for the system in Λ is defined as usual by setting $s(x)=+1$ for all $x\notin\Lambda$. Equivalently we take free boundary condition and we add a boundary external field acting on $x\in\partial_R\Lambda$,

$$h_b(x)=\sum_{y\notin\Lambda}J(x-y) \tag{3.8}$$

The existence of the thermodynamic limit for the + boundary condition is a well-known fact. We define a new boundary condition by replacing the boundary field (3.8) by the boundary field

$$h_b(x) = \lambda > 0, \qquad x \in \partial_R \Lambda \tag{3.9}$$

with λ arbitrarily small. We claim that the infinite volume equilibrium state $\langle \cdot \rangle^{\lambda}$ obtained from the boundary condition (3.9) is equal to the state $\langle \cdot \rangle^{+}$ obtained from + boundary condition. Indeed, if

$$\lambda \in G = \{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda,\ \mathrm{Re}\lambda > |\mathrm{Im}\,\lambda|\} \tag{3.10}$$

we have for

$$g_{\Lambda}(\lambda) = \frac{1}{1 + \langle s(x)\rangle_{\Lambda}^{\lambda}} \tag{3.11}$$

and

$$f_{\Lambda}(\lambda) = \frac{1}{1 + \langle s(x)s(y)\rangle_{\Lambda}^{\lambda}} \tag{3.12}$$

the bounds

$$|g_{\Lambda}(\lambda)| \leqslant 1, \qquad |f_{\Lambda}(\lambda)| \leqslant 1 \tag{3.13}$$

as consequences of (2.29) and (2.30). From the inequalities of Griffiths and for real $\lambda \geqslant \sum_x J(x)$ we have

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle s(x)\rangle_{\Lambda}^{\lambda} = \langle s(x)\rangle^{+} \tag{3.14}$$

and

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle s(x)s(y)\rangle_{\Lambda}^{\lambda} = \langle s(x)s(y)\rangle^{+} \tag{3.15}$$

Using the theorems of Vitali and Hurwitz as in the first application we get that $\lim_{\Lambda}\langle s(x)\rangle_{\Lambda}^{\lambda} = \langle s(x)\rangle^{\lambda}$ exists and is analytic in $\lambda \in G$, hence is equal to $\langle s(x)\rangle^{+}$ for $\lambda > 0$. Similarly $\langle s(x)s(y)\rangle^{\lambda} = \langle s(x)s(y)\rangle^{+}$ if $\lambda > 0$. Our assumption (3.6) implies that

$$\langle s(x)s(y)\rangle^{+} > 0, \qquad |x - y| = 1 \tag{3.16}$$

From this and the preceding results we obtain using the inequalities of Lebowitz[10] that $\langle \cdot \rangle^{+} = \langle \cdot \rangle^{\lambda}$. [We use (3.16) if $\langle s(x)\rangle^{+} = 0$, in order to prove that all odd correlation functions are zero and all even correlation functions are equal to those of $\langle \cdot \rangle^{+}$.]

Remark. Our second application concerning the boundary field is also valid for the planar rotor model since Dunlop proved in Ref. 9 the inequalities

$$\mathrm{Re}\langle \cos\theta(x)\rangle_{\Lambda}(\{h(x)\}) \geqslant 0 \tag{3.17}$$

and

$$\mathrm{Re}\langle\cos[\theta(x) \pm \theta(y)]\rangle_{\Lambda}(\{h(x)\}) \geqslant 0 \tag{3.18}$$

valid for $\mathrm{Re}\,h(x) > |\mathrm{Im}\,h(x)|$, where $h(x)$ is an external field

$$-\sum h(x)\cos\theta(x) \tag{3.19}$$

By arguing as above the result follows then from Ref. 3, which contains the generalization of Ref. 10 for this model.

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REFERENCES

1. F. Dunlop, *J. Stat. Phys.* **17**:215–228 (1977).
2. J. Ginibre, *Commun. Math. Phys.* **16**:310–328 (1970).
3. A. Messager, S. Miracle-Sole, and C. E. Pfister, *Commun. Math. Phys.* **58**:19–29 (1978).
4. J. L. Lebowitz and O. Penrose, *Commun. Math. Phys.* **11**:99–124 (1968).
5. J. L. Lebowitz, in *Quantum Dynamics Models and Mathematics*, ed. L. Streit (Springer, New York, 1976), pp. 201–220.
6. S. Miracle-Sole, *Colloquia Mathematica Societatis Janos Bolyai 27*, Random Fields, Esztergom (Hungary), pp. 767–786 (1979).
7. R. B. Griffiths, *J. Math. Phys.* **10**:1559–1565 (1969).
8. B. Simon and R. B. Griffiths, (1973). *Commun. Math. Phys.* **33**:145–164 (1973).
9. F. Dunlop, *J. Stat. Phys.* **21**:561–572 (1979).
10. J. L. Lebowitz, in *Mathematical Problems in Theoretical Physics* (Lecture Notes in Physics No. 80, Springer, New York, 1977), pp. 68–80.